Math 275D Lecture 1 Notes

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1 Introduction to Brownian Motion

1.1 Definition of Brownian motion

Here is a heuristic definition of Brownian motion, describing its properties.

Definition 1.1. Brownian motion is a "random" function $B(t)$ for $t \geq 0$, $t \in \mathbb{R}$ such that

1. $B(0) = 0$
2. (Independence on disjoint intervals) $B(t_2) - B(t_1)$ is independent of $B(t_4) - B(t_3)$ if $[t_3, t_4] \cap [t_2, t_1] = \varnothing$ for any $t_1, t_2, t_3, t_4 \in \mathbb{R}$ with $t_i \geq 0$.
3. For all $t > 0$, $B(t) \sim N(0, t)$.
4. $B(t)$ is continuous a.s.

We can ask many questions about Brownian motion. For example, we can ask what is $\sup_{0 \leq t \leq 1} |B(t)|$?

1.2 Comparison with the Poisson process

This is similar to a Poisson Process. Recall the Poisson Process:

Definition 1.2. A **Poisson process** is a "random" function $N(t)$ for $t \geq 0$, $t \in \mathbb{R}$ such that

1. $N(0) = 0$
2. $N(t_4) - N(t_3)$ is independent of $N(t_2) - N(t_1)$ if $[t_3, t_4] \cap [t_2, t_1] = \varnothing$ for any $t_1, t_2, t_3, t_4 \in \mathbb{R}$ with $t_i \geq 0$.
3. $N(t) \sim \mathrm{Pois}(t)$.

How do we know such a thing actually exists? We need to have a probability space $(\Omega, \mathcal{F}, \mathbb{P})$? What are the sample space and measurable sets we want to talk about? To talk about a Poisson process, we only need to know when the times of the jumps are. So we can take Ω to be the set of step functions and $\mathcal{F} = \sigma(N(t) : t \geq 0)$.

We can define $\Omega, \mathcal{F}$ for Brownian motion similarly. But how do we define $\mathbb{P}$? The Heuristic definition only defines $\mathbb{P}$ on some special events! Next time, we will make this rigorous. Later, we will define it in yet another, better way and show that the methods are equivalent.